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Primary Projections on L^2 of a Nilmanifold*

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Let N denote a connected, simply connected nilpotent Lie group with discrete cocompact subgroup Γ. Let U denote the quasi-regular representation on N on $L^2(N/\Gamma)$. $L^2(N/\Gamma)$ can be written as a direct sum of primary subspaces with respect to U. A realization for the projections of $L^2(N/\Gamma)$ onto these primary summands is given in this paper.

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Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Denote by $\mathfrak{n}^*$ the dual of $\mathfrak{n}$ and, by Ad the adjoint representation of N on $\mathfrak{n}$ and by Ad* the contragradient of the adjoint representation. In the Kirillov model for the unitary representations of N, there is a bijection between the equivalence classes of irreducible representations, $\hat{N}$, and the orbits of the contragradient representation in $\mathfrak{n}^*$. We will write π_ζ for the representation associated to the orbit in $\mathfrak{n}^*$ that contains the element ζ.

Suppose N contains a discrete cocompact subgroup Γ, and let U denote the quasi-regular representation of N on $L^2(N/\Gamma)$. Then $L^2(N/\Gamma)$ can be written as a direct sum of subspaces H_ζ, where U acts as a finite multiple of π_ζ on H_ζ. Denote by $(N/\Gamma)^\wedge$ the elements of $\hat{N}$ for which $H_\zeta \neq \{0\}$. One seeks the projection P_ζ of $L^2(N/\Gamma)$ onto H_ζ.

The elements occuring in $(N/\Gamma)^\wedge$ have been characterized by Howe [4] and Richardson [8]. A pair (χ, M), consisting of a subgroup M of N and a character χ on M, is called a *maximal character* if there is a $\zeta \in \mathfrak{n}^*$ and a subalgebra $\mathfrak{m}$ of $\mathfrak{n}$ such that (i) $\mathfrak{m}$ is subordinate to ζ, i.e., $\langle \zeta, [\mathfrak{m}, \mathfrak{m}] \rangle = 0$, (ii) $\mathfrak{m}$ has maximum dimension among the algebras subordinate to ζ, (iii) $M = \exp(\mathfrak{m})$, (iv) $\chi(m) = \exp 2\pi i \langle \zeta, \log m \rangle$ for all $m \in M$. A maximal character (χ, M) is called a *maximal integral character* if in addition, $M/\Gamma \cap M$ is compact and $\chi \mid \Gamma \cap M \equiv 1$. The Howe–Richardson occurence condition states that

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$\pi_\zeta \in (N/\Gamma)^\wedge$ if, and only if, there is a maximal integral character associated to some element in the orbit of ζ.

In [1], Corwin, Greenleaf, and Penny obtain a realization of P_ζ by first picking a rational $\zeta_0 \in o(\zeta)$ (the orbit of ζ in n^*) and a rational maximal subordinate subalgebra m such that $\langle \zeta_0, m \cap \log \Gamma \rangle \subseteq Z$. This gives rise to a maximal integral character (χ, M) where $\chi(m) = \exp(2\pi i \langle \zeta_0, \log m \rangle)$ for all $m \in M = \exp(m)$, with the properties that $\pi_\zeta = \operatorname{ind}(M \uparrow N, \chi)$ (see [1], [8]). They show that for smooth function f on N/Γ,

$$P_\zeta f(\Gamma) = \sum_{\gamma \in \Gamma/\Gamma \cap M} \int_{M^\gamma/\Gamma \cap M^\gamma} f(m) \overline{x(\gamma_m \gamma^{-1})}\, dm$$

where $M^\gamma = \gamma^{-1} M \gamma$.

In [2], Corwin, Greenleaf, and Penny obtain a canonical formula for P_ζ by replacing the rather arbitrarily chosen maximal subordinate subalgebra m by a subalgebra $k_\infty(\zeta)$ that is constructed in terms of ζ as follows (see [6]): let $r(\zeta) =$ radical (ζ), and let $k(\zeta)$ denote the ideal generated by $r(\zeta)$. Define $k_k(\zeta)$ inductively by $k_1(\zeta) = k(\zeta)$ and $k_{k+1}(\zeta) = k(\zeta \mid k_k(\zeta))$. Finally, let $k_\infty(\zeta) = \bigcap \{k_k(\zeta) \mid k = 1, 2, \ldots\}$. Then $k_\infty(\zeta)$ is a subalgebra subordinate to ζ, and hence ζ defines a character $\chi(\zeta)$ on $H_\infty(\zeta) = \exp k_\infty(\zeta)$ as usual. N acts on the pair $(\chi(\zeta), H_\infty(\zeta))$ by conjugation. A pair (χ', H') in the orbit of this action is called an integral point if $H'/H' \cap \Gamma$ is compact and $\chi' \mid \Gamma \cap H' \equiv 1$. Then, for $f \in C^\infty(N/\Gamma)$,

$$P_\zeta f(\Gamma) = \sum_{(\chi', H')} \int_{H'/H' \cap \Gamma} \chi'(h) f(h)\, dh,$$

the sum being over all integral points in the orbit of $(\chi(\zeta), H_\infty(\zeta))$.

In this paper a result is proven, that was suggested to this author by R. Howe (see also [3]), which allows for an alternate realization of the primary projections corresponding to orbits of maximal dimension.

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Let θ be a smooth function on n^* whose derivatives grow at most polynomially at infinity. θ may be viewed as a tempered distribution on n^*, and we denote its distributional Fourier transform by $\hat{\theta}$. Composition with the exponential map gives a tempered distribution in N, D_θ, i.e., for each $f \in S(N)$, the Schwartz space on N, we have

$$\begin{aligned}\langle D_\theta, f \rangle = \langle \hat{\theta}, \exp^* f \rangle &= \langle \theta, (\exp^* f)^\wedge \rangle \\ &= \int_{n^*} \theta(\zeta)(\exp^* f)^\wedge(\zeta)\, d\zeta,\end{aligned}$$

and $\exp^* f(X) = f(\exp X)$. Convolution by D_θ is defined on $S(N)$ by setting

$$D_\theta * f(n) = \langle D_\theta, f_n\rangle$$

for each $f \in S(N)$, $n \in N$ where $f_n(n') = f(n'n)$. It is shown in [3], that under the assumptions on θ, convolution by D_θ is a continuous endomorphism of $S(N)$. It follows that the map D'_θ, defined on $S'(N)$, the space of tempered distributions on N, by $\langle D'_\theta\phi, f\rangle = \langle\phi, D_\theta * f\rangle$ for each $\phi \in S'(N)$, $f \in S(N)$, is also a continuous endomorphism.

$L^2(N/\Gamma)$ is regarded as a subspace of $S'(N)$ by setting

$$\langle F, f\rangle = \int F(n)\, f(n)\, dn$$

for all $F \in L^2(N/\Gamma)$ and $f \in S(N)$.

THEOREM. *Let θ be a smooth,* Ad* N *invariant function on $\mathfrak{n}^*$ whose derivatives grow at most polynomially. Let Γ be a discrete cocompact subgroup of N and $\zeta \in \mathfrak{n}^*$ such that π_ζ occurs in $L^2(N/\Gamma)$. Let H_ζ be the primary summand of $L^2(N/\Gamma)$ corresponding to ζ. Then for $F \in H_\zeta$, $D'_\theta F = \theta(\zeta)F$.*

Remark. This result has been obtained by Kirillov [5] in the special case when Θ is a polynomial.

Proof. The proof will be by induction on the dimension of N.

If N is Abelian and $f \in S(N)$, then, identifying $\mathfrak{n}$ with N, we have

$$D_\theta * f(X) = \int \theta(\zeta)(f_X)\hat{}(\zeta)\, d\zeta = (\theta\hat{f})\hat{}(X).$$

H_ζ is the one dimensional subspace spanned by χ_ζ, the character corresponding to $\zeta \in \mathfrak{n}^*$, and

$$\begin{aligned}\langle D'_\theta\chi_\zeta, f\rangle &= \int \chi_\zeta(X)(\theta\hat{f})\hat{}(X)\, dX\\ &= \theta(\zeta)\hat{f}(\zeta)\\ &= \theta(\zeta)\langle\chi_\zeta, f\rangle,\end{aligned}$$

and hence the theorem is proved for Abelian groups.

Assume that the dimension of N exceeds one and that the dimension of the center of N equals one. Then there is an $X^* \in \mathfrak{n}^*$ such that $\theta(\eta + tX^*) = \theta(\eta)$ for all $\eta \in \mathfrak{n}^*$. To show this, let $\mathfrak{n} = \mathbb{R}X \oplus \mathfrak{n}_1$ where $\mathfrak{n}_1$ is a Kirillov subalgebra of $\mathfrak{n}$, and let X^* be dual to X in $\mathfrak{n}^*$. Then, if $\eta \in \mathfrak{n}^*$ and $\dim o(\eta)$ is maximum, $\mathbb{R}X^* + \eta' \subset o(\eta)$ for all $\eta' \in o(\eta)$, since $\mathrm{Ad}^* \exp tY \cdot \eta' = \eta' + t\langle\eta, Z\rangle X^*$, where $\mathfrak{n}_1$ is the centralizer of Y in $\mathfrak{n}$ and $Z = [X, Y]$ spans the center of $\mathfrak{n}$. Now note that the elements $\eta \in \mathfrak{n}^*$ with $\dim o(\eta)$ maximum is dense in $\mathfrak{n}^*$.

Let $N_0 = \exp(\mathbb{R}X)$ and $\Gamma_i = N_i \cap \Gamma$, $i = 0, 1$.

Fix a $\zeta \in n^*$ such that π_ζ occurs in U. Then by the Howe–Richardson multiplicity formulas we may assume that ζ gives rise to a maximal integral character (χ, M), i.e., there is a subalgebra, m, of maximal dimension subordinate to ζ such that, if $M = \exp(m)$, then $M/M \cap \Gamma$ is compact, and $\chi(\Gamma \cap M) = \{1\}$ where $\chi(m) = \exp 2\pi i\langle \zeta, \log m\rangle$ for $m \in M$. Furthermore, we may assume $M \subset N_1$.

Let $L_0^2(N/M, \chi)$ denote the measurable functions on N having compact support modulo M for which $f(nm) = \chi(m) f(n)$ for all $n \in N$ and $m \in M$ and with $\int_{N/M} |f(n)|^2\, dn < p$. Let B $(=B_{(\chi,M)})$ denote the mapping of $L_0^2(N/M, \chi)$ into H_χ given by

$$Bf(n\Gamma) = \sum_{\Gamma/\Gamma\cap M} f(n\gamma).$$

In [8], Richardson proved that B is a multiple of an isometry of $L_0^2(N/M, \chi)$ onto a dense subspace of an irreducible subspace of H_ζ. Furthermore, if $((M, \chi) \cdot N)_\#$ denotes the maximal integral characters among $\{(M^n, \chi^n) \mid n \in N\}$ $(M^n = n^{-1}Mn$ and $\chi^n(m) = \chi(nmn^{-1}))$ then

$$\bigcup B_{(M',\chi')}(L_0^2(N/M', \chi')), \qquad (M', \chi') \in ((M, \chi) \cdot N)_\#$$

is dense in H_ζ.

For $f \in S_0(N)$, the compactly supported elements of $S(N)$, and maximal integral character (M, χ), set

$$f *_M \chi(n) = \int_M f(ns)\, \chi(s^{-1})\, ds.$$

Then $f *_M \chi \in L_0^2(N/M, \chi)$ for each $f \in S_0(N)$. Let $S_s(N)$ denote the subset of all $f \in S_0(N)$ such that $f(n_0 n_1) = f_0(n_0) f_1(n_1)$ for all $n_i \in N_i$, and some $f_i \in S(N_i)$, $i = 0, 1$. The linear span of $S_s(N)$, $\langle S_s(N)\rangle$, is dense in $S_0(N)$, and

$$\bigcup B_{(M',\chi')}(\langle S_s(N)\rangle *_{M'} \chi), \qquad (M', \chi') \in ((M, \chi) \cdot N)_\#$$

is dense in H_ζ.

For $\eta \in n^*$ and $X \in n$, set $e(\eta \cdot X) = \exp 2\pi i\langle \eta, X\rangle$, and for $n \in N$, set $\bar{n} = \log n$. Note that using the $\mathrm{Ad}^* N$ invariance of θ, and the fact that the Jacobians of $\mathrm{Ad}^* n$ and $\mathrm{Ad}\, n$ are one, we have, for each $f \in S(N)$, and $n \in N$,

$$\begin{aligned}\langle D_\theta, f_n\rangle &= \int \theta(\eta) f(\exp(X)n)\, e(\eta \cdot X)\, dX\, d\eta \\ &= \int \theta(\eta) f(nn')\, e(\mathrm{Ad}^* n\eta \cdot \bar{n}')\, dn'\, d\eta \\ &= \int \theta(\eta) f(nn')\, e(\eta \cdot \bar{n}')\, dn'\, d\eta.\end{aligned}$$

Thus, identifying $L^2(N/\Gamma)$ with a subspace of $S'(N)$, we have, for $F \in L^2(N/\Gamma)$, $f \in S(N)$,

$$\begin{aligned}\langle D'_\theta F, f\rangle &= \int F(n)\langle D_\theta\, , f_n\rangle\, dn \\ &= \int F(n)\, f(nn')\, \theta(\eta)\, e(\eta \cdot \bar{n}')\, dn'\, d\eta\, dn.\end{aligned}$$

For $\eta \in \mathfrak{n}^*$ let η_0 and η_1 denote the components of η in $\mathbb{R}X^*$ and $\mathfrak{n}_1$ respectively. For $h \in S_s(N)$

$$\begin{aligned}\langle D_\theta\, , h\rangle &= \int h(n_0 n_1)\, \theta(\eta)\, e(\eta \cdot (n_0 n_1)^-)\, dn_0\, dn\ \ d\eta \\ &= \int h(n_0 n\ )\, \theta(\eta_0 + \eta\ ) \\ &\quad \times e((\eta_0 + \eta\ ) \cdot (\bar{n}_0 + \bar{n}_1 + \tfrac{1}{2}[\bar{n}_0\, , \bar{n}_1] + \cdots))\, dn_0\, dn_1\, d\eta_0\, d\eta_1 \\ &= \int h(n_0 n\ )\, \theta(\eta_1)\, e(\eta_0 \cdot \bar{n}_0)\, e(\eta_1 \cdot \bar{n}_1)\, e(\eta_1 \cdot (\tfrac{1}{2}[n_0\, , n_1] + \cdots))\, dn_0\, dn\ \ d\eta_0\, d\eta_1\end{aligned}$$

Now, $N_0 = \exp(\mathbb{R}X)$, and setting $E(t; n_1\, , \eta_1) = e(\eta_1 \cdot t(\tfrac{1}{2}[X, n_1] + \cdots))$, we have

$$\begin{aligned}&\int h_0(\exp tX)\, e(\eta_1 \cdot t(\tfrac{1}{2}[X, n_1] + \cdots))\, e(\eta_0 \cdot tX)\, dt\, d\eta_0 \\ &\qquad = \int (h_0 E(\cdot\,; n_1\, , \eta_1))^\wedge(\eta_0)\, d\eta_0 = h_0(e).\end{aligned}$$

Hence,

$$\langle D_\theta\, , h\rangle = \int h(n_1)\, \theta(\eta_1)\, e(\eta_1 \cdot \bar{n}_1)\, dn_1\, d\eta_1\, .$$

Assume now that $f, g \in S_s(N)$. Then

$$\begin{aligned}* &= \langle D'_\theta(B_{(M,\chi)}(f *_M \chi))\, g\rangle \\ &= \int B(f *_M \chi)(n)\, g(nn')\, \theta(\eta)\, e(\eta \cdot \bar{n}')\, dn'\, d\eta\, dn \\ &= \sum_{\gamma \in \Gamma/\Gamma \cap M} \int f(n\gamma m)\, \chi(m^{-1})\, g(nn')\, \theta(\eta)\, e(\eta \cdot \bar{n}')\, dm\, dn'\, d\eta\, dn \\ &= \sum_{\gamma_1 \in \Gamma_1/\Gamma_1 \cap M}\ \sum_{\gamma_0 \in \Gamma/\Gamma_1} \int f(n_0 \gamma_0 n_1^{\gamma_0} \gamma_1 m)\, \chi(m^{-1})\, g(n_0 n_1 n'_0 n'_1) \\ &\quad \times \theta(\eta)\, e(\eta \cdot \bar{n}')\, dm\, dn'_0\, dn'_1\, d\eta\, dn_0\, dn_1\end{aligned}$$

$$= \sum_{\gamma_0, \gamma_1} \int f_0(n_0\gamma_0) f_1(n_1^{\gamma_0}\gamma_1 m)\, \chi(m^{-1})\, g_0(n_0)\, g_1(n_1 n_1')$$

$$\times\, \theta(\eta_1)\, e(\eta_1 \cdot \bar{n}_1')\, dm\, dn'\, d\eta_1\, dn_0\, dn_1$$

$$= \sum_{\gamma_0} \int f_0(n_0\gamma_0)\, g_0(n_0) \int \sum_{\gamma_1} f_1 *_M \chi(n_1^{\gamma_0}\gamma_1)\, g_1(n_1 n_1')$$

$$\times\, \theta(\eta_1)\, e(\eta_1 \cdot \bar{n}_1')\, dm\, dn_1'\, d\eta_1\, dn_0\, dn_1 .$$

Let B_1 denote the mapping B defined with respect to the maximal integral character (M, χ) and N_1, and let θ_1 denote the restriction of θ to n_1^*. Then

$$* = \sum_{\gamma} \int f_0(n_0\gamma_0)\, g_0(n_0) \int B_1(f_1 *_M \chi)(n_1^{\gamma_0})\, D_{\theta_1} * g_1(n_1)\, dm\, dn_1'\, d\eta_1\, dn_0\, dn_1$$

$$= \sum_{\gamma_0} \int f_0(n_0\gamma_0)\, g_0(n_0) \langle D_{\theta_1}'(B_1(f_1 *_M \chi))^{\gamma_0}, g_1\rangle\, dn_0$$

$B_1(f_1 *_M \chi)$ is in the primary summand of $L^2(N_1/\Gamma_1)$ corresponding to the maximal integral character (M, χ). Thus, if $\zeta_1 \in n_1^*$ determines χ on M then $(B_1(f_1 *_M \chi))^{\gamma_0} \in H_{\mathrm{Ad}^*\gamma_0\zeta_1}$. By the induction hypothesis

$$D_{\theta_1}'(B_1(f_1 *_M \chi))^{\gamma_0} = \theta_1(\mathrm{Ad}^*\, \gamma_0\zeta_1)\, B_1(f_1 *_M \chi)^{\gamma_0}.$$

But $\theta_1(\mathrm{Ad}^*\, \gamma_0\zeta_1) = \theta_1(\zeta_1) = \theta(\zeta)$. Hence

$$* = \theta(\zeta) \sum_{\gamma_0} \int f_0(n_0\gamma_0)\, g_0(n_0) \langle B_1(f_1 *_M \chi)^{\gamma_0}, g_1\rangle\, dn_0$$

$$= \theta(\zeta) \sum_{\gamma_0, \gamma_1} \int f_0(n_0\gamma_0) f_1(n_1^{\gamma_0}\gamma_1 m)\, \chi(m^{-1})\, g_0(n_0)\, g_1(n_1)\, dm\, dn_1\, dn_0$$

$$= \theta(\zeta) \langle B(f *_M \chi), g\rangle.$$

The obvious density arguments can now be used to complete the proof under the current assumptions on N.

Suppose now that the center of n, $Z(n)$ has $\dim Z(n) \geqslant 2$ and that $\zeta \in n^*$ such that π_ζ occurs in $L^2(N/\Gamma)$. Let $\{X_n, ..., X_0\}$ be a Malcev basis for n with $\{X_1, X_0\} \subset Z(n)$ and $\langle \zeta, X_0\rangle = 0$. Let $W_i = \exp(\mathbb{R}X_i)$ and $N_1 = W_n \cdots W_1$. Let $n_1^* = X_0^{\perp}$ and let X_0^* be dual to X_0 in n^*, so that $n^* = n_1^* \oplus \mathbb{R}X_0^*$. For $\eta \in n^*$ let η_1 and η_0 denote the components of η in n_1^* and $\mathbb{R}X_0^*$ respectively. Assume that for fixed η_1, $\eta_0 \to \theta(\eta_0 + \eta_1) \in S(\mathbb{R})$.

If $F \in B_{(M,\chi)}(L_0^2(N/M, \chi))$ where (M, χ) is a maximal integral character corresponding to ζ, then

$$F(n_1 w) = \sum_{\gamma \in \Gamma/\Gamma \cap M} f(n_1 w\gamma) = \sum_{\gamma \in \Gamma/\Gamma \cap M} f(n_1 \gamma w) = F(n_1)$$

for $n_1 \in N_1$ and $w \in W_0$. Thus, if $g \in S(N)$ then

$$\langle D'_\theta F, g\rangle = \int F(n_1 w)\, g(n_1 n'_1 w w')\, \theta(\eta_0 + \eta_1)$$
$$\times\, e((\eta_0 + \eta_1) \cdot (\bar{n}'_1 + \bar{w}'))\, dw'\, dn'_1\, d\eta_0\, d\eta_1\, dw\, dn_1$$
$$= \int F(n_1)\, g((n_1 n'_1 w w')\theta)\, e(\eta_1 \cdot \bar{n}'_1)$$
$$\times\, \theta(\eta_0 + \eta_1)\, e(\eta_0 \cdot \bar{w}')\, dw'\, dn'_1\, d\eta_0\, d\eta_1\, dw\, dn_1 .$$

Since $W_0 \subset M$, and $\langle \zeta, X_0\rangle = 0$, $L_0{}^2(N/M, \chi) \equiv L_0{}^2((N/W_0)/(M/W_0), \chi)$. Also $\zeta = \zeta_1$. Hence $L^2(N/\Gamma) \supset H_\zeta = H_{\zeta_1} \subset L^2((N/W_0)/(\Gamma \cdot W_0/W_0))$. We consider ζ_1 as defined on $\mathfrak{n}/\mathbb{R}X_0$, the Lie algebra of N/W_0. For $g \in S(N)$, define $\bar{g}$ in $S(N/W_0)$ by

$$\bar{g}(n_1) = \int_{W_0} g(n_1 w)\, dw.$$

By the induction hypothesis,

$$\langle D'_\theta F, g\rangle = \int F(n_1)\, \bar{g}(n, n'_1)\, \theta(\eta_1)\, e(\eta_1 \cdot \bar{n}'_1)\, dn'_1\, d\eta_1\, dn_1$$
$$= \theta(\zeta_1)\langle F, \bar{g}\rangle$$
$$= \theta(\zeta)\langle F, g\rangle.$$

The assumption that $\eta_0 \to \theta(\eta_0 + \eta_1) \in S(\mathbb{R})$ was used to obtain the Fourier inversion

$$\theta(\eta_1)\, \bar{g}(n_1 n'_1) = \int g(n_1 n'_1 w w')\, \theta(\eta_0 + \eta_1)\, e(\eta_0 \cdot \bar{w}')\, dw'\, d\eta_0\, dw.$$

For more general θ observe that if we set $\phi_t(\eta) = \exp(-t\eta_0{}^2)$ and $\theta_t(\eta) = \phi_t(\eta)\, \theta(\eta)$, then $D'_{\theta_t}(\nu) = D'_{\phi_t}(D'_\theta(\nu))$ for all $\nu \in S'(N)$, and for all $f \in S(N)$, $D'_{\phi_t} * f \to f$ in $S(N)$ as $t \to 0$. Hence $D'_{\theta_t}(\nu) \to D'_\theta(\nu)$ in $S'(N)$. Since θ_t is $\mathrm{Ad}^* N$ invariant if θ is, and since $\eta_0 \to \theta_t(\eta_0 + \eta_1) \in S(\mathbb{R})$, the proof is complete.

COROLLARY. *Let θ be an* $\mathrm{Ad}^* N$ *invariant smooth function on n^* whose derivatives have polynomial growth at infinity. Then for $F \in L^2(N/\Gamma)$*

$$D'_\theta F = \sum_{\pi_\zeta \in (N\backslash\Gamma)^\wedge} \theta(\zeta)\, P_\zeta(F).$$

In order to obtain the aforementioned realization of P_ζ recall that by the Pukanszky parametrization of the orbits, [7], there is a Jordan–Hölder basis for $\mathfrak{n}^*$ and n ($=\dim \mathfrak{n}$) functions $\{R_j\,;\ 1 \leqslant j \leqslant n\}$ defined on $\mathbb{R}^d \times \mathbb{R}^{n-d}$

which are polynomials on $\mathbb{R}^d$ and rational functions on $\mathbb{R}^{n-d}$, and there is a Zariski open set Λ_0 in $\mathbb{R}^{n-d}$ such that for fixed $\lambda \in \Lambda_0$,

$$\left\{ \sum_{j=1}^{n} R_j(z, \lambda)\, e_j \;\middle|\; z \in \mathbb{R}^d \right\}$$

is an orbit in n^*, the orbits are distinct for distinct $\lambda \in \Lambda_0$, and the orbits of maximal dimension are all obtained in this manner.

If $\pi_\zeta \in (N/\Gamma)^\wedge$ and ζ has an orbit of maximum dimension then there is a $\lambda_\zeta \in \Lambda_0$ and a compact neighborhood U of λ_ζ in Λ_0 such that if $\pi_{\zeta'} \in (N/\Gamma)^\wedge$ then $\Lambda_{\zeta'} \notin U$. Let $\theta' \in C^\infty(\operatorname{int}(U))$ such that $\theta'(\lambda_\zeta) = 1$. Define θ on n^* by $\theta(\sum R_j(z, \lambda)\, e_j) = \theta'(\lambda)$ for each $\lambda \in \Lambda_0$ and let θ be zero elsewhere. Then θ is smooth, its derivatives are polynomially bounded, and it is $\mathrm{Ad}^* N$ invariant. Hence, for $F \in L^2(N/\Gamma)$, $D'_\theta F = \sum \theta(\eta) P_\eta(F) = P_\zeta(F)$ since $\theta(\zeta) = \theta(\sum R_j(0, \lambda_\zeta) e_j) = \theta'(\lambda_\zeta) = 1$ and $\theta(\eta) = 0$ for all other $\pi_\eta \in (N/\Gamma)^\wedge$. Thus $P_\zeta = D'_\theta$.

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